Hydraulic fracture of a porous thick-walled hollow sphere

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Background

Hydraulic fracture processes in permeable geomaterials with stationary fluid flow.

Aim

Study the importance of Biot coefficient for hydraulic fracture processes.

Methodology

Investigate fracture of permeable thick-walled hollow sphere subjected to inner fluid pressure.
Outline

Assumptions and notations

Fluid driven loading

Elastic response

Nonlinear fracture response

Size effect
Assumptions and notations

- Spherical symmetry
- Stationary flow
- Constant viscosity and permeability
- Small displacements
- Influence of gravity neglected

\( P_{fi} \) tension positive
Fluid driven loading
Fluid driven loading

\[ \dot{V} = \dot{V}_i + Q \quad \dot{V}_i = \dot{u}_i 4\pi r_i^2 \]

\[ q = \frac{Q}{4\pi r^2} \quad q = \frac{\kappa}{\mu} \frac{dP_f}{dr} \]

Pressure distribution

\[ \bar{P}_f = \bar{P}_{fi} \frac{\bar{r} - \bar{r}_o}{\bar{r} (1 - \bar{r}_o)} \]

with dimensionless variables

\[ \bar{P}_{fi} = P_{fi}/E \quad \bar{P}_f = P_f/E \quad \bar{r} = r/r_i \quad \bar{r}_o = r_o/r_i \]
Elastic response
Equilibrium condition

\[
\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_t}{r} = 0
\]

Stress definition

\[
\sigma_r = \sigma_r^m + bP_f
\]

\[
\sigma_t = \sigma_t^m + bP_f
\]

Constitutive laws

\[
\varepsilon_r = \frac{1}{E} \left( \sigma_r^m - 2\nu \sigma_t^m \right)
\]

\[
\varepsilon_t = \frac{1}{E} \left( (1 - \nu)\sigma_t^m - \nu\sigma_r^m \right)
\]

Kinematics

\[
\varepsilon_r = \frac{du}{dr} \quad \varepsilon_t = \frac{u}{r}
\]

Dimensionless ODE

\[
\frac{d^2 \bar{u}}{d \bar{r}^2} + 2 \frac{d \bar{u}}{d \bar{r}} \frac{1}{\bar{r}} - 2 \frac{\bar{u}}{\bar{r}^2} + b \bar{P}_f (1 + \nu) (1 - 2\nu) \frac{1}{(1 - \nu)} \frac{\bar{r}_o}{1 - \bar{r}_o} \frac{1}{\bar{r}^2} = 0
\]

with dimensionless radial displacement

\[
\bar{u} = \frac{u}{r_i}
\]

Solve analytically and numerically for boundary conditions:

\[
\bar{\sigma}_r^m = (1 - b) \bar{P}_f \text{ at } \bar{r} = \bar{r}_i
\]

\[
\bar{\sigma}_r^m = 0 \text{ at } \bar{r} = \bar{r}_o
\]
Radial displacement versus radius

\[ \nu = 0.2 \text{ and } \bar{r}_o = 7.25 \]

\[ \bar{r} \]
Radial stress versus radius

$$\sigma_r^m / P_{fi}$$

$b = 1$

$b = 0.5$

$b = 0$

$\nu = 0.2$ and $\bar{r}_o = 7.25$

Analytical

Numerical
Tangential stress versus radius

\[ \sigma_t \approx \frac{P}{2\pi} \left[ 1 - \nu \left( \frac{1}{r_o} - \frac{1}{\bar{r}_o} \right) \right] \]

\( \nu = 0.2 \) and \( \bar{r}_o = 7.25 \)
Nonlinear fracture response
Nonlinear fracture mechanics
Nonlinear fracture mechanics

Possible fracture pattern

\[ \bar{\sigma}_t^m = \varepsilon_0 \exp \left( -\frac{\varepsilon_t^c}{\varepsilon_f} \right) \text{ with } \varepsilon_0 = \frac{f_t}{E} \]

\[ \varepsilon_f = w_f \frac{l_c}{2S_c} = w_f \frac{\alpha}{r} = \bar{w}_f \frac{\alpha}{r} \text{ with } \bar{w}_f = \frac{w_f}{r_i} \]
Constitutive law for cracking

\[
\varepsilon_r = \frac{1}{E} (\sigma_r^m - 2\nu\sigma_t^m)
\]

\[
\varepsilon_t = \varepsilon_t^e + \varepsilon_t^c
\]

\[
\varepsilon_t^e = \frac{1}{E} \left( (1 - \nu) \sigma_t^m - \nu \sigma_r^m \right)
\]
Nonlinear ODE

\[
\frac{d^2 \bar{u}}{d\bar{r}^2} + 2 \frac{d \bar{u}}{d\bar{r}} \frac{1}{\bar{r}} - 2 \frac{\bar{u}}{\bar{r}^2} - \frac{2\nu}{1 - \nu} \frac{d \varepsilon_t^c}{d\bar{r}} + \frac{2(1 - 2\nu)}{1 - \nu} \frac{\varepsilon_t^c}{\bar{r}} + 
\]

\[
+b \frac{P_{fi}}{E} \frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)} \frac{\bar{r}_o}{1 - \bar{r}_o} \frac{1}{\bar{r}^2} = 0
\]

Numerical solution:

- Finite difference scheme
- Shooting method
- Newton method
- Outer displacement control
Pressure versus inner displacement

\[ \nu = 0.2, \; \bar{r}_o = 7.25 \; \text{and} \; \alpha \bar{w}_f = 0.01 \]
Tangential effective stress versus radius

\[ \frac{\dot{m}}{\dot{\varepsilon}_0} \]

\[ \sigma_t \]

\[ b=0 \quad \bar{r}_c = 3.08 \]

\[ b=1 \]

\[ 5.17 \]

\[ 7.25 \]

\[ \nu = 0.2, \quad \bar{r}_o = 7.25 \quad \text{and} \quad \alpha \bar{w}_f = 0.01 \]
$
u = 0.2$, $\bar{r}_o = 7.25$ and $\alpha \bar{w}_f = 0.01$
Size effect
Size effect

Investigate effect of inner radius $r_i$ on strength for constant $\bar{r}_o = r_o / r_i$

Dimensionless input affected by change of $r_i$:

$\bar{w}_f = \frac{w_f}{r_i}$ with $w_f = \text{const}$

Equilibrium

$\bar{P}_{fi} = -2 \int_{1}^{\bar{r}_o} (\bar{\sigma}_t^m + b\bar{P}_f) \bar{r} \, d\bar{r}$
**Limits**

**Plastic limit:**

\[ r_i \to 0 \Rightarrow \bar{w}_f = \frac{w_f}{r_i} \to \infty \]

\[ \frac{\overline{P}_{f_{i,pl}}}{(\bar{r}_o^2 - 1)} = -\frac{\varepsilon_0}{1 + b(\bar{r}_o - 1)} \]

**Onset of cracking:**

\[ r_i \to \infty \Rightarrow \bar{w}_f = \frac{w_f}{r_i} \to 0 \]

\[ \frac{\overline{P}_{f_{i,el}}}{\bar{r}_o^2 - 1} = -\frac{1}{\bar{r}_o^2 - 1} \left( \frac{2\varepsilon_0}{1 - b\frac{1 - 2\nu}{1 - \nu}} \right) \left( \frac{\bar{r}_o^3 + 2}{\bar{r}_o^3 - 1} \right) + b\frac{1}{1 - \nu} \frac{\bar{r}_o - 2\nu}{\bar{r}_o - 1} \]
Size effect for $\bar{r}_o = 7.25$

$-\bar{P}_{fi,\text{max}}/(\bar{r}_o^2 - 1)$

$\bar{P}_{fi,\text{pl}}/(\bar{r}_o^2 - 1)$

$\bar{P}_{fi,\text{el}}/(\bar{r}_o^2 - 1)$

$\nu = 0.2$

$\alpha \bar{w}_f = 0.01$

$\alpha \bar{w}_f = 0.02$

$\alpha \bar{w}_f = 0.04$

$\alpha \bar{w}_f = 0.08$
Size effect for $\bar{r}_o = 3.125$

$-\bar{P}_{f_i,\text{max}}/\left(\bar{r}_o^2 - 1\right)$

$\bar{P}_{f_i,\text{pl}}/\left(\bar{r}_o^2 - 1\right)$

$\bar{P}_{f_i,\text{el}}/\left(\bar{r}_o^2 - 1\right)$

$\nu = 0.2$
Size effect for $\bar{r}_o = 14.5$

\[ \frac{-P_{f_i, \text{max}}}{(\bar{r}_o^2 - 1)} \]

\[ \frac{\bar{P}_{f_i, \text{pl}}}{(\bar{r}_o^2 - 1)} \]

\[ \frac{\bar{P}_{f_i, \text{el}}}{(\bar{r}_o^2 - 1)} \]

\[ \nu = 0.2 \]
Conclusions

- Model for fluid driven fracture in a thick-walled sphere based on nonlinear fracture mechanics
- Strong effect of Biot-coefficient on strength
- Strong effect of size on strength, which decreases with increasing Biot-coefficient and decreasing thickness of the sphere.