Hydraulic fracture of a permeable thick-walled hollow sphere

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Background

Hydraulic fracture processes in permeable geomaterials (concrete, rock, stiff soils) with stationary fluid flow.

Aim

Study the importance of Biot coefficient for hydraulic fracture processes.

Methodology

Investigate initial fracture (hardening) of permeable thick-walled hollow sphere subjected to inner fluid pressure.
Assumptions and notations
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- Spherical symmetry
- Stationary flow
- Constant viscosity and permeability
- Fluid is incompressible
- Small displacements
- Influence of gravity neglected

$P_{f_i}$ tension positive
Fluid driven loading
Fluid driven loading

\[ \dot{V} = \dot{V}_i + Q \quad \dot{V}_i = \dot{u}_i 4\pi r_i^2 \]

\[ q = \frac{Q}{4\pi r^2} \quad q = \frac{\kappa}{\mu} \frac{dP_f}{dr} \]

Pressure distribution

\[ P_f = \frac{\bar{P}_f}{P_f} \frac{\bar{r} - \bar{r}_o}{\bar{r} (1 - \bar{r}_o)} \]

with dimensionless variables

\[ \bar{P}_f = P_f / E \quad \bar{P}_i = P_i / E \quad \bar{r} = r / r_i \quad \bar{r}_o = r_o / r_i \]
Elastic response
Equilibrium condition

\[
\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_t}{r} = 0
\]

Stress definition

\[
\sigma_r = \sigma_r^m + bP_f
\]
\[
\sigma_t = \sigma_t^m + bP_f
\]

Constitutive laws

\[
\varepsilon_r = \frac{1}{E} \left( \sigma_r^m - 2\nu \sigma_t^m \right)
\]
\[
\varepsilon_t = \frac{1}{E} \left( (1 - \nu) \sigma_t^m - \nu \sigma_r^m \right)
\]

Kinematics

\[
\varepsilon_r = \frac{d\varepsilon_t}{dr} ~ \varepsilon_t = \frac{u}{r}
\]

Dimensionless ODE

\[ \frac{d^2 \bar{u}}{d\bar{r}^2} + 2 \frac{d\bar{u}}{d\bar{r}} \frac{1}{\bar{r}} - 2 \frac{\bar{u}}{\bar{r}^2} + b\bar{P}_f \frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)} \frac{\bar{r}_o}{1 - \bar{r}_o} \frac{1}{\bar{r}^2} = 0 \]

with dimensionless radial displacement

\[ \bar{u} = \frac{u}{r_i} \]

Solve analytically and numerically for boundary conditions:

\[ \bar{\sigma}_r^m = (1 - b)\bar{P}_f \text{ at } \bar{r} = \bar{r}_i \]

\[ \bar{\sigma}_r^m = 0 \text{ at } \bar{r} = \bar{r}_o \]
Radial displacement versus radius

-\tilde{u}/P_{fi} \quad \text{vs} \quad \bar{r}

\nu = 0.2 \quad \text{and} \quad \bar{r}_{o} = 7.25

b = 0, 0.5, 1

Analytical, Numerical
Nonlinear fracture response
Nonlinear fracture mechanics

Possible fracture pattern

\[ \bar{\sigma}_t^m = \varepsilon_0 \exp \left( -\frac{\varepsilon_t^c}{\varepsilon_f} \right) \quad \text{with} \quad \varepsilon_0 = \frac{f_t}{E} \]

\[ \varepsilon_f = w_f \frac{l_c}{2S_c} = w_f \frac{\alpha}{r} = \bar{w}_f \frac{\alpha}{\bar{r}} \quad \text{with} \quad \bar{w}_f = \frac{w_f}{r_i} \]
Constitutive law for cracking

\[ \varepsilon_r = \frac{1}{E} \left( \sigma_r^m - 2\nu \sigma_t^m \right) \]

\[ \varepsilon_t = \varepsilon_t^e + \varepsilon_t^c \]

\[ \varepsilon_t^e = \frac{1}{E} \left( (1 - \nu) \sigma_t^m - \nu \sigma_r^m \right) \]
Nonlinear ODE

\[
\frac{d^2 \bar{u}}{d\bar{r}^2} + 2 \frac{d\bar{u}}{d\bar{r}} \frac{1}{\bar{r}} - 2 \frac{\bar{u}}{\bar{r}^2} - \frac{2\nu}{1-\nu} \frac{d\varepsilon_t^c}{d\bar{r}} + \frac{2(1-2\nu)}{1-\nu} \frac{\varepsilon_t^c}{\bar{r}} + \\
+ b \frac{P_{fi}}{E} \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \frac{\bar{r}_o}{1-\bar{r}_o} \frac{1}{\bar{r}^2} = 0
\]

Equilibrium:

\[
\bar{P}_{fi} = -2 \int_1^{\bar{r}_o} (\bar{\sigma}_t^m + b\bar{P}_f) \bar{r} \, d\bar{r}
\]

Show \( \frac{\bar{P}_{fi}}{\bar{r}_o^2 - 1} \) versus \( \bar{u}_i \)
Pressure versus inner displacement

\[-P_{fi}/(\bar{r}_o^2 - 1)\] vs. \[\bar{u}_i\]

\[\nu = 0.2, \bar{r}_o = 7.25 \text{ and } \alpha \bar{w}_f = 0.01\]
\[\nu = 0.2, \quad \bar{r}_o = 7.25 \quad \text{and} \quad \alpha \bar{w}_f = 0.01\]
Size effect
Size effect

Investigate effect of inner radius $r_i$ on strength for constant $\bar{r}_o = r_o/r_i$

Dimensionless input affected by change of $r_i$:

$$\bar{w}_f = \frac{w_f}{r_i} \text{ with } w_f = \text{const}$$
Limits

Plastic limit:

\[ r_i \to 0 \implies \bar{w}_f = \frac{w_f}{r_i} \to \infty \]

\[ \frac{\bar{P}_{fi,pl}}{(\bar{r}_o^2 - 1)} = -\frac{\varepsilon_0}{1 + b(\bar{r}_o - 1)} \]

Onset of cracking:

\[ r_i \to \infty \implies \bar{w}_f = \frac{w_f}{r_i} \to 0 \]

\[ \frac{\bar{P}_{fi,el}}{\bar{r}_o^2 - 1} = -\frac{1}{\bar{r}_o^2 - 1} \left( 1 - b \frac{1 - 2\nu}{1 - \nu} \right) \frac{2\varepsilon_0}{\bar{r}_o^3 + 2} + b \frac{1}{1 - \nu} \frac{\bar{r}_o - 2\nu}{\bar{r}_o - 1} \]
Size effect for $\bar{r}_o = 7.25$

$\alpha \bar{w}_f = 0.01$

$\alpha \bar{w}_f = 0.02$

$\alpha \bar{w}_f = 0.04$

$\alpha \bar{w}_f = 0.08$

$-\bar{P}_{f_{i, max}}/(\bar{r}_o^2 - 1)$

$\bar{P}_{f_{i, pl}}/(\bar{r}_o^2 - 1)$

$\bar{P}_{f_{i, el}}/(\bar{r}_o^2 - 1)$

$\nu = 0.2$

$b$
Conclusions

- Model for fluid driven fracture in a thick-walled sphere based on nonlinear fracture mechanics
- Strong effect of Biot-coefficient on strength
- Strong effect of size on strength, which decreases with increasing Biot-coefficient and decreasing thickness of the sphere.