Nonlocal plastic models for cohesive-frictional materials

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Two classes of nonlocal models for softening materials with nonassociated plastic flow are described and their localization properties are analyzed. Models of the first class are based on standard plasticity and use a hardening-softening law dependent on both local and nonlocal cumulative plastic strain. Models of the second class are based on combination of plasticity and damage mechanics and use a damage evolution law dependent again on local and nonlocal cumulative plastic strain. Conditions for continuous and discontinuous bifurcations from a uniform state in an infinite medium are derived, and the evolution of localized plastic zone is illustrated by simple one- and two-dimensional examples.

1 INTRODUCTION

Concrete is a cohesive-frictional material with a complex behavior. A realistic description of its constitutive law in the framework of plasticity requires nonassociated flow rules and transition from hardening to softening under tension or low-confined compression. Both nonassociated flow and softening are destabilizing phenomena that can lead to the loss of ellipticity of the governing differential equations. The static boundary value problem then becomes ill-posed and the numerical solution suffers from pathological sensitivity to the discretization parameters, e.g., to the element size in the finite element method.

Mathematical and numerical problems related to the loss of ellipticity can be overcome by suitable regularization techniques, e.g., by a nonlocal formulation of the constitutive model. In nonlocal continua, stress at a point depends not only on the strain and temperature history at that point but in general on the history of the entire body. Since the influence of remote points should be very small, in practice it is neglected and the nonlocal interaction is restricted to points at distance closer than a characteristic length R, called the interaction radius.

In nonlocal formulations of the integral type, a certain state variable (usually an internal variable) is replaced by its nonlocal counterpart, obtained by weighted averaging over a neighborhood of radius R of each material point. Such formulations have been well developed for damage models; see the pioneering paper by Pijaudier-Cabot and Bažant (1987) and the comparative study by Jirásek (1998). For plasticity models, regularization techniques have often been based on nonlocal formulations of the differential (gradient) type. Nevertheless, a number of integral-type plasticity models exist as well; they have been systematically compared and evaluated by Jirásek and Rolshoven (2003). It turns out that some of these models have certain drawbacks, e.g., they do not preclude localization of plastic strain to a set of zero measure, or cannot properly reflect the entire softening process up to complete failure.

The purpose of this paper is to present two different approaches to integral-type regularization of nonassociated softening plasticity. Formulations with nonlocal hardening laws are addressed in Section 2. An alternative approach, applicable to combined plastic-damage models and based on a nonlocal damage law, is considered in Section 3. For both cases, analysis of bifurcations from a uniform state in an unbounded medium is performed and localization conditions are derived. Illustrative examples in one and two dimensions are presented. The results serve as a basis for the development of a fully-fledged nonlocal model for failure of concrete.

2 BASIC EQUATIONS OF PLASTICITY AND DAMAGE MECHANICS

To facilitate the discussion of nonlocal models, it is useful to start with a brief review of the basic equations of plasticity and damage mechanics in the context of the standard "local" continuum. We consider the flow theory of plasticity with a single smooth yield surface, and a simple isotropic damage theory with one scalar damage parameter. Attention is restricted to small strain and to rate-independent behavior.

In the small-strain theory of **plasticity**, the total strain $\boldsymbol{\varepsilon}$ is additively decomposed into the elastic part, $\boldsymbol{\varepsilon}_e$, and the plastic part, $\boldsymbol{\varepsilon}_p$. The stress $\boldsymbol{\sigma}$ is then linked by the linear elastic law

$$\boldsymbol{\sigma} = \boldsymbol{D}_e : \boldsymbol{\varepsilon}_e = \boldsymbol{D}_e : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) \tag{1}$$

to the elastic part of strain. Here, stress and strain are considered as second-order tensors, the elastic properties are reflected by the fourth-order elastic stiffness tensor $\boldsymbol{D}_e$, and the colon denotes the tensorial operation of double contraction. The evolution of the plastic strain is described by the flow rule

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}\boldsymbol{g}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{2}$$

in which $\boldsymbol{g}$ is a given function specifying the direction of plastic flow, and $\dot{\lambda} \geq 0$ is the rate of the plastic multiplier, to be determined from additional conditions. Dot over a symbol denotes differentiation with respect to time, but time plays here only the role of a formal parameter that controls the loading process. Equation (2) and other rate equations to follow could alternatively be presented in an incremental form, with rates replaced by infinitesimal increments.

Plastic flow can take place only if the stress attains a critical level that may depend on certain variables $\boldsymbol{q}$. For simplicity, we consider only isotropic hardening, controlled by a single variable κ. The yield function is postulated in the form

$$f(\boldsymbol{\sigma}, \kappa) = \sigma_{eq}(\boldsymbol{\sigma}) - \sigma_{\mathrm{Y}}(\kappa) \tag{3}$$

where σ_{eq} is the equivalent stress and σ_{Y} is the yield stress. The yield function is designed such that states with $f < 0$ are elastic, plastic yielding can take place only if $f = 0$, and $f > 0$ is impossible. This is expressed by the loading-unloading conditions

$$\dot{\lambda} \geq 0, \qquad f(\boldsymbol{\sigma}, \kappa) \leq 0, \qquad \dot{\lambda} f(\boldsymbol{\sigma}, \kappa) = 0 \tag{4}$$

The dependence of the yield stress σ_{Y} on the hardening variable κ is called the hardening law; we write it here in the form

$$\sigma_{\mathrm{Y}}(\kappa) = \sigma_0 + h(\kappa) \tag{5}$$

where σ_0 is the initial yield stress and h is the hardening function, vanishing for $\kappa = 0$. The hardening variable κ is usually related to the evolution of plastic strain, e.g. through the strain-hardening or work-hardening hypothesis. In general, it can be defined by the rate equation

$$\dot{\kappa} = \dot{\lambda} k(\boldsymbol{\sigma}, \kappa) \tag{6}$$

where parameter k may depend on the stress and hardening variable, but often is just a constant scaling factor.

By an appropriate choice of the expression for equivalent stress $\sigma_{eq}(\boldsymbol{\sigma})$ and of the scaling factor $k(\boldsymbol{\sigma}, \boldsymbol{\kappa})$, it is possible to endow the equivalent stress and the hardening variable with the desired physical meaning. For instance, σ_{eq} and κ can be scaled such that, under uniaxial stress along axis x, the equivalent stress is equal to the normal stress σ_{xx} and the hardening variable is (during monotonic loading) equal to the plastic strain $\varepsilon_{p,xx}$. The hardening function can then be directly identified from the uniaxial stress-strain curve. The derivative of the hardening function, $H = \mathrm{d}h/\mathrm{d}\kappa$, is called the plastic modulus. If $H > 0$, the yield stress increases and the material is said to be hardening. The opposite case $H < 0$ is referred to as softening, and in the limit case $H = 0$ we say that the behavior is perfectly plastic.

The typical format of the stress-strain law used by the scalar **damage** model reads

$$\boldsymbol{\sigma} = (1-\omega)\boldsymbol{D}_e : \boldsymbol{\varepsilon} \tag{7}$$

where ω is the damage parameter whose value ranges between 0 in the virgin (undamaged state) and 1 in the fully damaged state. The growth of the damage parameter must be described by a suitable evolution equation. It is convenient to use a damage law in the explicit form

$$\omega = g_d(\kappa_d) \tag{8}$$

where κ_d is another internal variable, which physically corresponds to the maximum strain level reached in the previous history of the material point. This is formally described by the loading-unloading conditions

$$\dot{\kappa}_d \geq 0, \qquad f_d(\boldsymbol{\varepsilon}, \kappa_d) \leq 0, \qquad \dot{\kappa}_d f_d(\boldsymbol{\varepsilon}, \kappa_d) = 0 \tag{9}$$

in which the damage loading function f_d is defined as

$$f_d(\boldsymbol{\varepsilon}, \kappa_d) = \varepsilon_{eq}(\boldsymbol{\varepsilon}) - \kappa_d \tag{10}$$

and ε_{eq} is a scalar measure of strain, called the equivalent strain.

3 PLASTICITY WITH NONLOCAL HARDENING LAW

3.1 *Nonlocal hardening law*

Consider a plastic model with softening, i.e., with decreasing yield stress at increasing cumulative plastic strain. It is easy to show that, for the local version of the model and in the one-dimensional case, plastic strain can localize in one point of the interval that represents the one-dimensional model of a bar under tension or of a semi-infinite layer under shear. Failure then occurs at vanishing energy dissipation, which is physically inadmissible.

The model response can be regularized by a nonlocal formulation with the yield stress dependent not only on the "local" value of the hardening variable κ but also on its nonlocal average

$$\bar{\kappa}(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{x}, \boldsymbol{s})\kappa(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} \tag{11}$$

Here, $\alpha(\boldsymbol{x}, \boldsymbol{s})$ is a nonlocal weight function that describes the strength of interaction between points $\boldsymbol{x}$ and $\boldsymbol{s}$ and decays with increasing distance between these points. The weight function is usually nonnegative and normalized such that $\int_V \alpha(\boldsymbol{x}, \boldsymbol{s})\,\mathrm{d}\boldsymbol{s} = 1$ for all $\boldsymbol{x} \in V$.

The type of nonlocal formulation considered here consists in replacing the hardening law (5) by

$$\sigma_{\mathrm{Y}}(\kappa, \bar{\kappa}) = \sigma_0 + h^*(\kappa, \bar{\kappa}) \tag{12}$$

where h^* is a given function which for $\kappa = \bar{\kappa}$ reduces to the original hardening function h. For further developments, it is convenient to rewrite (12) in the rate form

$$\dot{\sigma}_{\mathrm{Y}} = H_L\dot{\kappa} + H_{NL}\dot{\bar{\kappa}} \tag{13}$$

in which $H_L = \partial h^*/\partial\kappa$ and $H_{NL} = \partial h^*/\partial\bar{\kappa}$ are the local and nonlocal plastic modulus, resp. In general, both H_L and H_{NL} can be functions of κ and $\bar{\kappa}$. Specific choices of the generalized hardening function will be discussed in Section 3.4.

3.2 *Rate problem*

To get insight into the regularizing effect of the nonlocal formulation, we analyze bifurcations from a uniform state of an infinite medium. Attention is restricted to special localization modes, for which planes parallel to a given reference plane remain planar, and the stress and strain rates depend only on the distance from the reference plane. Physically, such a case corresponds to a straight shear band or fracture process zone in a large body, with negligible influence of boundaries.

The reference plane is defined by its normal $\boldsymbol{n}$. Let us introduce a local Cartesian coordinate system with axis ξ in the direction of $\boldsymbol{n}$ and axes η and ζ perpendicular to it. The current state is considered as uniform, i.e., the current values of stress $\boldsymbol{\sigma}$, strain $\boldsymbol{\varepsilon}$ and all other variables do not depend on the spatial coordinates. The rates to be found, $\dot{\boldsymbol{\sigma}}(\xi)$, $\dot{\boldsymbol{\varepsilon}}(\xi)$ etc., are functions of the local coordinate ξ.

The loading program is defined by a suitable combination of strain and stress rates that are controlled on the remote boundaries. We will look for localized solutions with plastic yielding limited to a band centered around $\xi = 0$ and with the material far from the band elastically unloading. In this case, the far-field stress and strain rates are linked by the invertible elastic law, and mixed conditions can be easily transformed into conditions for the strain rates or the stress rates. Therefore, we can assume that the far-field conditions are given by

$$\lim_{\xi \to \pm\infty} \dot{\boldsymbol{\varepsilon}}(\xi) = \dot{\mu}\boldsymbol{e} \tag{14}$$

$$\lim_{\xi \to \pm\infty} \dot{\boldsymbol{\sigma}}(\xi) = \dot{\mu}\boldsymbol{s} \tag{15}$$

where $\boldsymbol{e}$ is a given reference strain, $\boldsymbol{s} = \boldsymbol{D}_e : \boldsymbol{e}$ is the corresponding reference stress, and $\dot{\mu}$ is an undetermined load multiplier rate.

From compatibility equations it follows that the in-plane strain rate components $\dot{\varepsilon}_{\eta\eta}$, $\dot{\varepsilon}_{\zeta\zeta}$ and $\dot{\varepsilon}_{\eta\zeta}$ must be uniform, independent of ξ. This is formally expressed by the equation

$$\dot{\boldsymbol{\varepsilon}}(\xi) = \dot{\mu}\boldsymbol{e} + [\boldsymbol{n} \otimes \dot{\boldsymbol{m}}(\xi)]_{sym} \tag{16}$$

where $\boldsymbol{m}$ is the polarization vector. From equilibrium equations it follows that the out-of-plane stress rate components $\dot{\sigma}_{\xi\xi}$, $\dot{\sigma}_{\xi\eta}$ and $\dot{\sigma}_{\xi\zeta}$ must be uniform, independent of ξ. This is formally expressed by the equation

$$\boldsymbol{n} \cdot \dot{\boldsymbol{\sigma}}(\xi) = \dot{\mu}\boldsymbol{n} \cdot \boldsymbol{s} \tag{17}$$

where $\boldsymbol{n} \cdot \boldsymbol{s} = \boldsymbol{n} \cdot \boldsymbol{D}_e : \boldsymbol{e}$ is the reference far-field traction.

Substituting the flow rule (2) into the rate form of the stress-strain law (1), we obtain the expression for the stress rate

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \boldsymbol{g}\dot{\lambda}) \tag{18}$$

in terms of the strain rate $\dot{\boldsymbol{\varepsilon}}$ and the rate of the plastic multiplier $\dot{\lambda}$. Since the current state is assumed to be given and to be uniform, the flow direction $\boldsymbol{g}$ can be treated as a known constant. The current value of the yield function must also be constant, and it must be zero, otherwise the material response would remain elastic as long as $f < 0$ and no localization would be possible. If $f = 0$, its rate $\dot{f}$ must satisfy the complementarity conditions

$$\dot{\lambda} \geq 0, \qquad \dot{f} \leq 0, \qquad \dot{\lambda}\dot{f} = 0 \tag{19}$$

that are similar to (4) but with f replaced by $\dot{f}$. Differentiating (3) and using (13) and (6), we can express the rate of the yield function

$$\dot{f} = \boldsymbol{f} : \dot{\boldsymbol{\sigma}} - k(H_L\dot{\lambda} + H_{NL}\dot{\bar{\lambda}}) \tag{20}$$

as a function of the stress rate and of the rate of the plastic multiplier. The gradient of the yield function with respect to the stress, $\boldsymbol{f} = \partial f / \partial \boldsymbol{\sigma}$, can also be considered as a given constant (under the assumption of a uniform state).

The problem to be solved consists in relating the rates $\dot{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{\varepsilon}}_p$, $\dot{\boldsymbol{\sigma}}$, $\dot{\lambda}$ and $\dot{\boldsymbol{m}}$ to the load rate $\dot{\mu}$. All these rates are in general functions of ξ, while the current values $\boldsymbol{f}$, $\boldsymbol{g}$, H_L and H_{NL} are constant and their values are known, same as the reference values $\boldsymbol{e}$ and $\boldsymbol{s}$. The orientation vector $\boldsymbol{n}$ is an arbitrary unit vector and the question to be answered is whether there exists a certain $\boldsymbol{n}$ for which the rate problem admits a nonuniform solution with the plastic zone localized into a band of finite width and whether this nonuniform solution can be discontinuous.

3.3 *Localization analysis*

Since the averaging operator in (20) acts on the plastic multiplier rate, we select $\dot{\lambda}(x)$ as the primary unknown field and eliminate all the other rates from the problem. First, combining (16)–(18) we obtain

$$\boldsymbol{n}\cdot\boldsymbol{D}_e : \left[\dot{\mu}\boldsymbol{e} + [\boldsymbol{n}\otimes\dot{\boldsymbol{m}}(\xi)]_{sym} - \boldsymbol{g}\dot{\lambda}(\xi)\right] = \dot{\mu}\boldsymbol{n}\cdot\boldsymbol{s} \tag{21}$$

Since $\boldsymbol{s} = \boldsymbol{D}_e : \boldsymbol{e}$, the terms $\dot{\mu}\boldsymbol{n}\cdot\boldsymbol{D}_e : \boldsymbol{e}$ and $\dot{\mu}\boldsymbol{n}\cdot\boldsymbol{s}$ cancel out. Due to minor symmetry of the elastic stiffness tensor, we have

$$\boldsymbol{n}\cdot\boldsymbol{D}_e : [\boldsymbol{n}\otimes\dot{\boldsymbol{m}}(\xi)] = (\boldsymbol{n}\cdot\boldsymbol{D}_e\cdot\boldsymbol{n})\cdot\boldsymbol{m}(\xi) = \boldsymbol{Q}_e\cdot\boldsymbol{m}(\xi) \tag{22}$$

where

$$\boldsymbol{Q}_e = \boldsymbol{n}\cdot\boldsymbol{D}_e\cdot\boldsymbol{n} \tag{23}$$

is the elastic acoustic tensor, known to be invertible. We can therefore use (21) to express $\dot{\boldsymbol{m}}$ in terms of $\dot{\lambda}$ and then substitute into the expressions (16), (18) and (20) for the strain rate, stress rate and rate of the yield function:

$$\dot{\boldsymbol{m}}(\xi) = \boldsymbol{Q}_e^{-1}\cdot(\boldsymbol{n}\cdot\boldsymbol{D}_e : \boldsymbol{g})\dot{\lambda}(\xi) \tag{24}$$

$$\dot{\boldsymbol{\varepsilon}}(\xi) = \dot{\mu}\boldsymbol{e} + \left[\boldsymbol{n}\otimes\boldsymbol{Q}_e^{-1}\cdot(\boldsymbol{n}\cdot\boldsymbol{D}_e : \boldsymbol{g})\right]_{sym}\dot{\lambda}(\xi) \tag{25}$$

$$\dot{\boldsymbol{\sigma}}(\xi) = \dot{\mu}\boldsymbol{D}_e : \boldsymbol{e} + \left[(\boldsymbol{D}_e\cdot\boldsymbol{n})\cdot\boldsymbol{Q}_e^{-1}\cdot(\boldsymbol{n}\cdot\boldsymbol{D}_e : \boldsymbol{g}) - \boldsymbol{D}_e : \boldsymbol{g}\right]\dot{\lambda}(\xi) \tag{26}$$

$$\dot{f}(\xi) = \dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e} + kH_{crit}^{(\boldsymbol{n})}\dot{\lambda}(\xi) - k\left[H_L\dot{\lambda}(\xi) + H_{NL}\dot{\bar{\lambda}}(\xi)\right] \tag{27}$$

In the last expression,

$$H_{crit}^{(\boldsymbol{n})} = k^{-1}\left[(\boldsymbol{f} : \boldsymbol{D}_e\cdot\boldsymbol{n})\cdot\boldsymbol{Q}_e^{-1}\cdot(\boldsymbol{n}\cdot\boldsymbol{D}_e : \boldsymbol{g}) - \boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{g}\right] \tag{28}$$

is the critical value of the plastic modulus that would allow, for the given orientation $\boldsymbol{n}$, a discontinuous bifurcation; see Rudnicky and Rice (1975) or Runesson et al. (1991). With $\dot{f}$ given by (27), the complementarity conditions (19) depend exclusively on the plastic multiplier rate, its nonlocal average, and the load multiplier rate. The problem to be solved is an integral version of a linear complementarity problem.

Let us look for solutions with plastic flow localized in an interval I_p. Inside I_p, the plastic multiplier rate is positive and the rate of the yield function must vanish. The condition $\dot{f} = 0$ with $\dot{f}$ given by (27) has the character of a Fredholm integral equation of the form

$$A\dot{\bar{\lambda}}(\xi) + B\dot{\lambda}(\xi) = C, \qquad \xi\in I_p \tag{29}$$

with

$$A = kH_{NL}, \qquad B = k(H_L - H_{crit}^{(\boldsymbol{n})}), \qquad C = \dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e} \tag{30}$$

The solution $\dot{\lambda}(\xi)$ is admissible if it is nonnegative in I_p and if the corresponding rate of the yield function, $\dot{f}(\xi)$, is nonpositive for $\xi \notin I_p$. Since $\dot{\lambda}(\xi)$ vanishes for $\xi \notin I_p$, this latter condition can be written as

$$A\dot{\bar{\lambda}}(\xi) \geq C, \qquad \xi \notin I_p \tag{31}$$

If $B \neq 0$, we can express from (29)

$$\dot{\lambda}(\xi) = \frac{C - A\dot{\bar{\lambda}}(\xi)}{B}, \qquad \xi \in I_p \tag{32}$$

The nonlocal plastic multiplier rate $\dot{\bar{\lambda}}(\xi)$ is continuous by construction. Consequently, the local rate $\dot{\lambda}(\xi)$ satisfying (32) is also continuous inside I_p. Consider a point ξ_{ep} on the boundary of I_p, separating the localized plastic zone from the surrounding zone of elastic unloading (or of neutral loading). As ξ approaches ξ_{ep} from the interior of I_p, $\dot{\lambda}(\xi)$ tends to its limit value

$$\dot{\lambda}_{ep} = \frac{C - A\dot{\bar{\lambda}}(\xi_{ep})}{B} \tag{33}$$

which must be nonnegative. Since $\dot{\bar{\lambda}}$ is continuous, its value at ξ_{ep} must satisfy condition (31), i.e., the numerator in (33) is nonpositive. Now if $B > 0$, the fraction on the right-hand side of (33) is nonpositive, and condition $\dot{\lambda}_{ep} \geq 0$ can be satisfied only if $\dot{\lambda}_{ep}$ vanishes. In this case, the rate of the plastic multiplier is continuous not only inside the plastic zone but everywhere, because outside the plastic zone it vanishes by definition and the transition between the elastic and plastic zones has just been proven to be continuous.

The foregoing mathematical analysis has shown that if $B > 0$, i.e., if

$$H_L > H_{crit}^{(\boldsymbol{n})} \tag{34}$$

then the solution must be continuous. Physically, this means that if the "local part" of the plastic modulus is above the critical plastic modulus determined by the standard localization analysis based on the acoustic tensor for the given orientation $\boldsymbol{n}$, the plastic strain rate cannot develop a discontinuity across a plane with normal $\boldsymbol{n}$. The maximum value of $H_{crit}^{(\boldsymbol{n})}$ over all possible orientations will be denoted as H_{crit}. If

$$H_L > H_{crit} \equiv \max_{\boldsymbol{n}} H_{crit}^{(\boldsymbol{n})} \tag{35}$$

then the plastic strain rate must remain continuous (provided that discontinuities across planes are the most critical ones, which is only assumed here, but not rigorously proven). Note that, in general, both sides of the inequality in (35) depend on the current state of the material, in particular on the stress and the hardening variables. If a certain model satisfies condition (35) for all states that can be attained, discontinuous bifurcations (of the planar type considered here) cannot take place and the model can be considered as properly regularized.

A suitable nonlocal formulation should surpress discontinuous localization but, at the same time, should allow the development of continuous localization patterns. Let us check under which conditions the rate problem can be expected to have a (nontrivial, i.e., nonuniform) solution. Two particular properties of the nonlocal averaging operator related to averaging of nonnegative fields turn out to be useful. First, the nonlocal average can nowhere exceed the maximum value of the local field. Second, if the local field vanishes on one side of a plane and is strictly positive at least in a small but finite band on the other side, then the gradient of the nonlocal field evaluated on that plane is oriented to the side on which the local field is positive.

These two properties will now be applied to the nonlocal plastic multiplier rate. Let us return to equation (32). We assume that $B > 0$, so that discontinuities are excluded. Since $\dot{\lambda}(\xi) \geq 0$, we obtain

$$A\dot{\bar{\lambda}}(\xi) \leq C, \qquad \xi \in I_p \tag{36}$$

On the elastoplastic boundary, we have $A\dot{\bar{\lambda}}(\xi_{ep}) = C$. As point ξ moves from ξ_{ep} into the interior of I_p, $\dot{\bar{\lambda}}(\xi)$ is increasing (due to the second property) but $A\dot{\bar{\lambda}}(\xi)$ must not increase, so as to satisfy (36). This is possible only if $A \le 0$. Since $\dot{\bar{\lambda}}(\xi_{ep}) > 0$ and $C = A\dot{\bar{\lambda}}(\xi_{ep})$, we can also show that $C \le 0$. Knowing this, we can exploit the first property and write

$$\max_{\xi \in I_p} \dot{\bar{\lambda}}(\xi) \le \max_{\xi \in I_p} \dot{\lambda}(\xi) \tag{37}$$

Using (32) and $B > 0$, we obtain

$$(A + B) \max_{\xi \in I_p} \dot{\bar{\lambda}}(\xi) \le C \tag{38}$$

and since $C \le 0$ and $\dot{\bar{\lambda}}(\xi) \ge 0$, we conclude that $A + B$ must be nonpositive. If this condition is not satisfied, no solution of the rate problem with plastic strain rate localized into a finite interval can exist. The condition $A + B \le 0$ means that

$$H_L + H_{NL} \le H_{crit}^{(\boldsymbol{n})} \tag{39}$$

Obviously, conditions (34) and (39) can be satisfied simultaneously only if the "nonlocal part" of the plastic modulus, H_{NL}, is negative.

To complete the theoretical analysis, consider the case when $B \le 0$, i.e., $H_L \le H_{crit}^{(\boldsymbol{n})}$. In this case, a discontinuous bifurcation can be expected. Indeed, suppose that the local rate $\dot{\lambda}(\xi)$ localizes into a narrow band of thickness h, much smaller than the nonlocal interaction radius R. The nonlocal rate $\dot{\bar{\lambda}}(\xi)$ is then almost constant in I_p and, according to (32), the local rate is also almost constant. Let us therefore set $\dot{\lambda}(\xi) = \dot{\lambda} =$ const. for $\xi \in I_p$ and $\dot{\lambda}(\xi) = 0$ for $\xi \notin I_p$. For convenience, we place the plastic zone at $\xi = 0$. The nonlocal rate is given by $\dot{\bar{\lambda}}(\xi) = \dot{\lambda} h \alpha_0(\xi)$ where function α_0 is defined by $\alpha_0(\xi) = \alpha(0, \xi)$; note that this function is nonnegative and has its maximum at $\xi = 0$. Equation (29) is rewritten as

$$A\dot{\lambda}h\alpha_0(0) + B\dot{\lambda} = C \tag{40}$$

and condition (31) as

$$A\dot{\lambda}h\alpha_0(\xi) \ge C, \qquad \xi \ne 0 \tag{41}$$

Combining this and taking into account that $\dot{\lambda} > 0$, we get

$$Ah[\alpha_0(0) - \alpha_0(\xi)] \le -B, \qquad \xi \ne 0 \tag{42}$$

The term $\alpha_0(0) - \alpha_0(\xi)$ varies between 0 and $\alpha_0(0)$, and h is very small. Consequently, if B is positive, condition (42) cannot be satisfied. This is consistent with our previous result that no discontinuities can develop for $B > 0$. If $B = 0$, condition (42) is satisfied if and only if $A \le 0$. Finally, if $B < 0$, condition (42) is satisfied independently of the value of A (because even if $A > 0$, it suffices to chose h smaller than $-B/[A\alpha_0(0)]$).

It is worth noting that C is proportional to the load parameter rate, $\dot{\mu}$. If $B = 0$ and $A = 0$, the load rate vanishes, i.e., localized yielding takes place at constant load (and constant stress and strain outside the plastic zone). If $B = 0$ and $A < 0$, the load rate is proportional to the product $\dot{\lambda}h$, which is proportional to the rate of displacement jump across the plastic zone. Physically, this means that the nonlocal model gives the same solution as a cohesive zone model based on a traction-separation law. The plastic zone is localized into a set of zero measure, but the dissipation does not vanish and the load-displacement diagram on the structural level does not suffer by pathological sensitivity to the mesh size. Finally, if $B < 0$, the load rate is proportional to the plastic strain rate. Since the plastic zone is arbitrarily thin, the dissipation rate becomes negligible and the solution is physically meaningless.

In summary, five types of localization patterns can be expected, depending on the values of local and nonlocal plastic moduli:

- $H_L \geq H_{crit}$ and $H_L + H_{NL} > H_{crit}$ ... no localization
- $H_L > H_{crit}$ and $H_L + H_{NL} \leq H_{crit}$ ... localization into a band, continuous strain rate
- $H_L = H_{crit}$ and $H_L + H_{NL} = H_{crit}$ ... localization into a plane, yielding at constant load
- $H_L = H_{crit}$ and $H_L + H_{NL} < H_{crit}$ ... localization into a plane, similar to cohesive zone
- $H_L < H_{crit}$ ... localization into a plane, vanishing dissipation

3.4 *Examples*

The simplest form of nonlocal hardening law is obtained if, in the local law (5), κ is replaced by its nonlocal average $\bar{\kappa}$. In other words, function h^* in (12) is defined as $h^*(\kappa, \bar{\kappa}) = h(\bar{\kappa})$. The local plastic modulus H_L then vanishes and condition (35) is satisfied if the critical plastic modulus is negative. In the one-dimensional setting, we have $H_{crit} = 0$. This means that localization takes place if $H_{NL} \leq 0$, and the plastic strain localizes into a set of zero measure (a point in the one-dimensional case) but the dissipation does not vanish. Such a formulation is essentially equivalent to a cohesive zone model.

Full regularization, with a nonvanishing size of the localized plastic region, can be achieved if the softening process is driven by a combination of the local and nonlocal cumulative plastic strains,

$$\hat{\kappa} = m\bar{\kappa} + (1-m)\kappa \tag{43}$$

in which m is a new model parameter. The nonlocal hardening function is given by

$$h^*(\kappa, \bar{\kappa}) = h(m\bar{\kappa} + (1-m)\kappa) \tag{44}$$

and the local and nonlocal plastic moduli are

$$H_L = (1-m)H, \qquad H_{NL} = mH \tag{45}$$

where $H = \mathrm{d}h/\mathrm{d}\hat{\kappa}$ in general depends on $\hat{\kappa}$. This type of nonlocal formulation was first proposed by Vermeer and Brinkgreve (1994) and later analyzed by Planas et al. (1996) and used by Strömberg and Ristinmaa (1996). All of them considered h as a decreasing function, i.e., $H < 0$. In this case, condition (35) can be satisfied if m is sufficiently larger than 1. For instance, in the one-dimensional case or for associated flow (i.e., for $\boldsymbol{g} = \boldsymbol{f}$), we have $H_{crit} = 0$ and it is sufficient to choose $m > 1$. The nonlocal plastic modulus $H_{NL} = mH$ is then always negative. In the one-dimensional setting, plastic strain localizes right away at the onset of yielding into a finite interval, the size of which is proportional to the interaction radius R and depends also on the parameter m (tends to zero for $m \to 1^+$ and tends to infinity for $m \to \infty$). The size of the plastic zone does not change during softening, same as the shape of the plastic strain profile. For $m = 1$, the model reduces to the previously discussed simple nonlocal model with hardening law $\sigma_{\mathrm{Y}} = \sigma_0 + h(\bar{\kappa})$. For $m < 1$, condition (35) could easily be violated and the energy dissipation for the fully localized solution would vanish.

The nonlocal formulation of the Vermeer-Brinkgreve type with $m > 1$ can serve as a localization limiter for plastic models with softening. However, if softening is preceded by hardening, we have $H > 0$ and $H_L = (1-m)H < 0$. The effect of the nonlocal term then becomes destabilizing and leads to localization of plastic strains. Therefore, this type of nonlocal formulation has a limited scope and cannot be used for general elastoplastic models (at least not with fixed m). The assumption that softening starts immediately at the elastic limit would be too artificial for materials like concrete, if general stress paths (including compression) are to be covered.

Reasonable response in both regimes, hardening and softening, can be obtained with a modified form of nonlocal hardening law, in which the current yield stress is expressed as a product of two terms that depend respectively on the local and nonlocal cumulative plastic strain:

$$\sigma_{\mathrm{Y}}(\kappa, \bar{\kappa}) = h_L(\kappa) h_{NL}(\bar{\kappa}) \tag{46}$$

Existing models of this type are closely related to the concept of ductile damage. The local part of the hardening law can be interpreted as the effective yield stress, i.e., as the yield stress of the

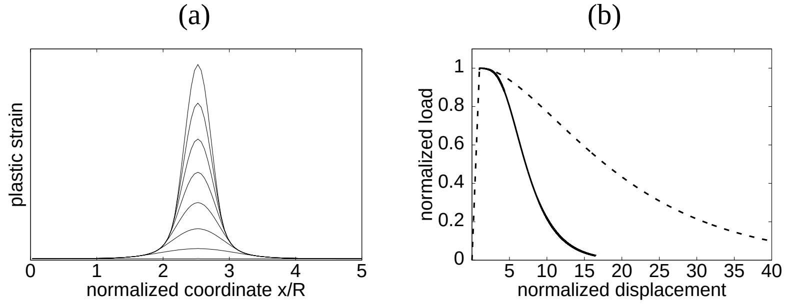

Figure 1: Ductile damage model of Geers and coworkers with exponential damage law: (a) evolution of plastic strain profile, (b) load-displacement diagram.

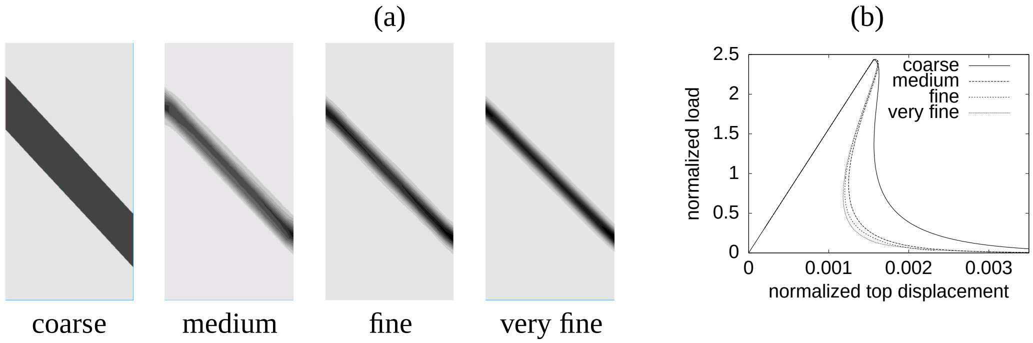

Figure 2: Localization into a shear band simulated with the nonlocal Drucker-Prager model on different finite element meshes: (a) distribution of the local softening variable κ, (b) load-displacement diagrams.

bulk material between voids and cracks, and the nonlocal part as the reduction due to damage. This concept was used for metals by Geers et al. (2001), who considered the local part as linear, $h_L(\kappa) = \sigma_0 + H_0\kappa$, where $H_0 > 0$ is the hardening modulus of the metallic matrix. The nonlocal part was written in the form $h_{NL}(\bar{\kappa}) = 1 - \omega_p(\bar{\kappa})$ with ω_p = ductile damage variable reflecting the influence of voids and varying between 0 and 1. Depending on the choice of the function relating ω_p to $\bar{\kappa}$, various shapes of the stress-strain diagram can be obtained. The local and nonlocal plastic moduli are given by

$$H_L = \frac{\mathrm{d}h_L}{\mathrm{d}\kappa} h_{NL} = (1 - \omega_p) H_0 \tag{47}$$

$$H_{NL} = \frac{\mathrm{d}h_{NL}}{\mathrm{d}\bar{\kappa}} h_L = -(\sigma_0 + H_0\kappa)\omega_p' \tag{48}$$

where $\omega_p' = \mathrm{d}\omega_p/\mathrm{d}\bar{\kappa}$. If $\omega_p < 1$ and $H_0 > 0$, the local plastic modulus is positive. The ductile damage model by Geers et al. (2001) uses the von Mises yield condition with an associated flow rule, for which the maximum critical hardening modulus is zero. Consequently, positiveness of the matrix hardening modulus H_0 is sufficient to preclude the discontinuous bifurcation. The nonlocal plastic modulus H_{NL} is always negative, and if its magnitude is large enough, plastic strain can localize into a band of finite thickness. The evolution of the plastic strain profile in a one-dimensional localization test is shown in Fig. 1a and the load-displacement diagram in Fig. 1b. For comparison, the diagram corresponding to the (unstable) uniform solution is plotted as the dashed curve.

In Rolshoven (2003), hardening law of the form (46) was adapted for materials like concrete. In applications to cohesive-frictional materials, one must in general admit nonassociated flow rules, and the critical hardening modulus can become positive. Still, if the local plastic modulus is kept sufficiently large, the discontinuous bifurcation cannot appear (at least not under the idealized conditions of infinite medium under uniform state and discontinuity along a plane).

The nonlocal model with hardening law (46) and with a sufficiently large value of the local plastic modulus does not allow discontinuous bifurcation, but localization is still possible. The localized plastic region has a certain minimum width and energy dissipation does not vanish. This is illustrated in Fig. 2a, which shows a shear band in a rectangular specimen under plane strain compression, captured by the finite element method on four different meshes. In this academic example, we use the Drucker-Prager yield function and a nonassociated flow rule, i.e., $\boldsymbol{g} \neq \boldsymbol{f}$. Upon mesh refinement, the load-displacement diagram converges and no pathological mesh dependence is observed; see Fig. 2b.

Plasticity models with dependence of the yield stress on the nonlocal cumulative plastic strain can provide an objective description of the localization process. However, this type of nonlocal formulation requires a complicated stress-return algorithm due to coupling among Gauss points. This motivates the search for an alternative regularization technique that would lead to a less expensive numerical algorithm.

4 PLASTICITY WITH NONLOCAL DAMAGE LAW

4.1 *Combined plastic-damage model*

For materials like concrete, the standard plasticity theory with dissipation only by plastic yielding and with no degradation of the elastic stiffness can be considered only as a rough approximation. This type of description is appropriate under highly confined compression, when the propagation of cracks is restricted and the deformation process indeed has a plastic character. Under tensile loading, but also under unconfined compression or shear, cracking plays an important role and has a major influence on the failure process. Still, formally it is possible to use a plastic model with softening, but from the physical point of view it is more appropriate to take the cracks into account by reduction of the elastic stiffness, in the spirit of continuum damage mechanics. Since a pure damage model would again have a limited scope, the most appropriate modeling framework seems to be the combination of plasticity theory and damage mechanics.

A natural generalization of stress-strain laws (1) and (7) that contains both of them as special cases is

$$\boldsymbol{\sigma} = (1-\omega)\boldsymbol{D}_e : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) = (1-\omega)\tilde{\boldsymbol{\sigma}} \tag{49}$$

Here, the effective stress $\tilde{\boldsymbol{\sigma}} = \boldsymbol{D}_e : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p)$ can be interpreted as the actual stress in the undamaged material between voids and cracks, and $1-\omega$ is the reduction factor that transforms the effective stress into the nominal stress, transmitted by the material on the macroscopic level. In addition to the stress-strain law, we need to specify which quantities should be used in the evolution equations for internal variables such as $\boldsymbol{\varepsilon}_p$, κ or ω. Here, the choice is not unique. For instance, the yield condition can be written for the effective stress or for the nominal one. Damage can be driven by the total strain, elastic strain, or plastic strain. The formulation should take into account the physical origin of inelastic processes in the particular material, but also the implications for the numerical implementation, regularization technique, etc.

The model proposed here is based on yield condition for effective stress and damage driven by cumulative plastic strain. So the yield function is written as

$$f(\tilde{\boldsymbol{\sigma}}, \kappa) = \sigma_{eq}(\tilde{\boldsymbol{\sigma}}) - \sigma_{\mathrm{Y}}(\kappa) \tag{50}$$

and the internal variable κ_d that drives the evolution of damage is set equal to the cumulative plastic strain κ, i.e., damage is evaluated as $\omega = g_d(\kappa)$. The yield stress σ_{Y} is assumed to be a nondecreasing function of κ but it represents the yield value of the equivalent *effective* stress. If yielding takes place at constant or slowly increasing effective stress, the *nominal* stress can decrease due to the growth of damage. In this way, softening is described as the combined effect of plastic flow and propagation of defects.

In the nonlocal version of the damage-plastic model, the variable κ_d that drives damage is determined as a combination of the local and nonlocal cumulative plastic strain. To make sure that κ_d will never decrease, we define the damage loading function

$$f_d(\kappa_d, \kappa, \bar{\kappa}) = m\bar{\kappa} + (1-m)\kappa - \kappa_d \tag{51}$$

and enforce the complementarity conditions (9). Damage is still evaluated explicitly as $\omega = g_d(\kappa_d)$. If $f_d = 0$, the damage rate can be expressed as

$$\dot{\omega} = g_d' \dot{\kappa}_d = g_d' \left\langle m\dot{\bar{\kappa}} + (1-m)\dot{\kappa} \right\rangle = g_d' k \left\langle m\dot{\bar{\lambda}} + (1-m)\dot{\lambda} \right\rangle \tag{52}$$

where $g_d' = \mathrm{d}g_d/\mathrm{d}\kappa_d$ and $\langle \dots \rangle$ is the positive-part operator.

4.2 *Rate problem*

The problem of bifurcation from a uniform state in an infinite medium will now be reformulated for the combined plastic-damage model. The general approach is the same as in Section 3.2, but eqs. (18) and (20) must be replaced by

$$\dot{\boldsymbol{\sigma}} = (1-\omega)\dot{\tilde{\boldsymbol{\sigma}}} - \tilde{\boldsymbol{\sigma}}\dot{\omega} = (1-\omega)\boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \boldsymbol{g}\dot{\lambda}) - \tilde{\boldsymbol{\sigma}} g_d' k \left\langle m\dot{\bar{\lambda}} + (1-m)\dot{\lambda} \right\rangle \tag{53}$$

and

$$\dot{f} = \boldsymbol{f} : \dot{\tilde{\boldsymbol{\sigma}}} - H\dot{\kappa} = \boldsymbol{f} : \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \boldsymbol{g}\dot{\lambda}) - kH\dot{\lambda} \tag{54}$$

Substituting (53) and (16) into (17) and using the relation $\boldsymbol{s} = (1-\omega)\boldsymbol{D}_e : \boldsymbol{e}$, we obtain an equation from which we can express

$$\dot{\boldsymbol{m}}(\xi) = \boldsymbol{Q}_e^{-1} \cdot (\boldsymbol{n} \cdot \boldsymbol{D}_e : \boldsymbol{g})\dot{\lambda}(\xi) + \boldsymbol{Q}_e^{-1} \cdot \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \frac{g_d' k}{1-\omega} \left\langle m\dot{\bar{\lambda}}(\xi) + (1-m)\dot{\lambda}(\xi) \right\rangle \tag{55}$$

This is the modified form of (24). Substituting (55) into (16) and (54), we get the rate of the yield function in terms of the rate of load multiplier and local and nonlocal rates of plastic multiplier,

$$\dot{f}(\xi) = \dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e} + k\left(H_{crit,p}^{(\boldsymbol{n})} - H\right)\dot{\lambda}(\xi) + k\Delta H_{crit,d}^{(\boldsymbol{n})} \left\langle m\dot{\bar{\lambda}}(\xi) + (1-m)\dot{\lambda}(\xi) \right\rangle \tag{56}$$

in which

$$H_{crit,p}^{(\boldsymbol{n})} = k^{-1}(\boldsymbol{f} : \boldsymbol{D}_e \cdot \boldsymbol{n}) \cdot \boldsymbol{Q}_e^{-1} \cdot (\boldsymbol{n} \cdot \boldsymbol{D}_e : \boldsymbol{g}) \tag{57}$$

is the critical value of plastic modulus evaluated for the plasticity model only, and

$$\Delta H_{crit,d}^{(\boldsymbol{n})} = H_{crit}^{(\boldsymbol{n})} - H_{crit,p}^{(\boldsymbol{n})} = \frac{g_d'}{1-\omega}(\boldsymbol{f} : \boldsymbol{D}_e \cdot \boldsymbol{n}) \cdot \boldsymbol{Q}_e^{-1} \cdot \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \tag{58}$$

is the difference between the critical modulus of the combined plastic-damage model and the critical modulus of the plasticity model; this difference is attributed to the presence of the damage term.

4.3 *Localization analysis*

The consistency condition $\dot{f} = 0$, to be satisfied inside the plastic zone, has again the character of a Fredholm integral equation, but this time it contains a damage-related term that must not be negative and therefore is placed in the positive-part brackets. If damage grows everywhere inside the plastic zone, these brackets can be omitted and the integral equation has the form (29) with parameters

$$A = -mk\Delta H_{crit,d}^{(\boldsymbol{n})}, \;\; B = k\left(H - H_{crit,p}^{(\boldsymbol{n})}\right) + (m-1)k\Delta H_{crit,d}^{(\boldsymbol{n})}, \;\; C = \dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e} \tag{59}$$

Based on the general results derived in Section 3.3, we expect a continuous bifurcation if $B > 0$ and $A + B < 0$, i.e., if

$$H_{crit,p}^{(\boldsymbol{n})} + (1-m)\Delta H_{crit,d}^{(\boldsymbol{n})} < H < H_{crit,p}^{(\boldsymbol{n})} + \Delta H_{crit,d}^{(\boldsymbol{n})} \equiv H_{crit}^{(\boldsymbol{n})} \tag{60}$$

and a discontinuous bifurcation if $B \leq 0$, i.e., if

$$H \leq H_{crit,p}^{(\boldsymbol{n})} + (1-m)\Delta H_{crit,d}^{(\boldsymbol{n})} \tag{61}$$

However, we must also admit that the combination $m\dot{\bar{\lambda}}(\xi) + (1-m)\dot{\lambda}(\xi)$ could be negative. Obviously, this can happen only inside the plastic zone and only if $m > 1$ (or $m < 0$, which is not considered here). In this case, the last term in (56) vanishes and the consistency condition

$$\dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e} + k\left(H_{crit,p}^{(\boldsymbol{n})} - H\right)\dot{\lambda}(\xi) = 0, \qquad \xi \in I_p = (-h/2, h/2) \tag{62}$$

is not an integral equation but a simple algebraic equation. Since $\dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e} \leq 0$ and $\dot{\lambda}(\xi) > 0$, condition (62) can be satisfied only if $H \leq H^{(\boldsymbol{n})}_{crit,p}$.

Consider first the critical case $H = H^{(\boldsymbol{n})}_{crit,p}$. According to (62), $\dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e}$ must vanish and $\dot{\lambda}(\xi)$ can be arbitrary. The solution is admissible only if $\dot{f} \leq 0$ outside the plastic zone, which is equivalent to $km\Delta H^{(\boldsymbol{n})}_{crit,d}\dot{\bar{\lambda}}(\xi) \leq 0$ for $\xi \notin I_p$. Sufficiently close to the elastoplastic boundary, the nonlocal rate $\dot{\bar{\lambda}}(\xi)$ is strictly positive, and so the condition can be satisfied only if $\Delta H^{(\boldsymbol{n})}_{crit,d} \leq 0$. Since the term $(\boldsymbol{f} : \boldsymbol{D}_e \cdot \boldsymbol{n}) \cdot \boldsymbol{Q}_e^{-1} \cdot \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{n}$ in (58) is usually positive (but this must be verified for each particular model), localization of the present type is excluded if $g'_d > 0$, i.e., if the damage growth is activated. If this is not the case, the model is reduced to a local plasticity model with a critical value of hardening modulus, and a classical discontinuous bifurcation occurs.

Let us now turn attention to the case that H is strictly smaller than $H^{(\boldsymbol{n})}_{crit,p}$. The rate of the plastic multiplier $\dot{\lambda}(\xi)$ solved from (62) is then uniform (independent of ξ) and equal to

$$\dot{\lambda} = \frac{\dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e}}{k\left(H - H^{(\boldsymbol{n})}_{crit,p}\right)} \tag{63}$$

If the plastic zone has a width h much smaller than the nonlocal interaction radius R, the nonlocal rate of plastic multiplier is given by

$$\dot{\bar{\lambda}}(\xi) = \dot{\lambda}h\alpha_0(\xi) = \frac{\dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e}h\alpha_0(\xi)}{k\left(H - H^{(\boldsymbol{n})}_{crit,p}\right)} \tag{64}$$

and condition $\dot{f} \leq 0$ outside the plastic zone reads

$$\dot{\mu}\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{e} + km\Delta H^{(\boldsymbol{n})}_{crit,d}\dot{\bar{\lambda}}(\xi) \leq 0, \qquad \xi \notin I_p \tag{65}$$

This is easily satisfied if h is chosen sufficiently small, and so the case $H < H^{(\boldsymbol{n})}_{crit,p}$ leads to a discontinuous bifurcation independently of the value of $\Delta H^{(\boldsymbol{n})}_{crit,d}$. The plastic zone is fully localized and damage grows only in the neighborhood of the plastic zone but not in the plastic zone itself.

In summary, localization analysis of the present type of nonlocal plastic-damage model shows that the behavior is governed by two critical values of plastic modulus, H_{crit} and $H_{crit,p}$, which are both obtained by classical localization analysis of the corresponding local model based on the acoustic tensor, but H_{crit} is evaluated for the complete plastic-damage model while $H_{crit,p}$ is evaluated for the plastic part only, with damage growth disabled. Bifurcation from a uniform state can be expected if the actual plastic modulus H drops below H_{crit}. If H remains above $H_{dis} = H_{crit,p} + \langle 1 - m \rangle \Delta H_{crit,d}$, the strain rate remains continuous. Discontinuous bifurcation occurs if H is smaller than H_{dis}. In the limit case $H = H_{dis}$, the bifurcation is continuous for $m > 1$ and discontinuous for $m < 1$. For $m = 1$, the bifurcation is discontinuous but the energy dissipation does not vanish and the behavior is similar to a cohesive zone model.

4.4 *Examples*

The main results are first illustrated by a one-dimensional example. In one dimension, the critical plastic modulus for the plastic model without damage is $H_{crit,p} = 0$, and the critical plastic modulus for the combined plastic-damage model $H_{crit} = \tilde{\sigma}g'_d/(1 - \omega)$ corresponds to a vanishing tangent stiffness of the combined model. Three parameter combinations are considered, and the results are plotted in Fig. 3 in terms of load-displacement diagrams and graphs showing the evolution of plastic strain and damage. For $H = 0$ and $m = 1$, plastic strain is fully localized and damage is distributed in an interval of width $2R$. For $H = 0$ and $m = 2$, plastic strain is distributed in a continuous manner and damage is uniform in the plastic zone and decreases continuously to zero in intervals of width R around the plastic zone. For $H > 0$ and $m = 1$, both plastic strain and damage are continuously distributed, with maximum values in the middle of the localized zone. In all three cases, the load-displacement diagrams converge upon mesh refinement and the

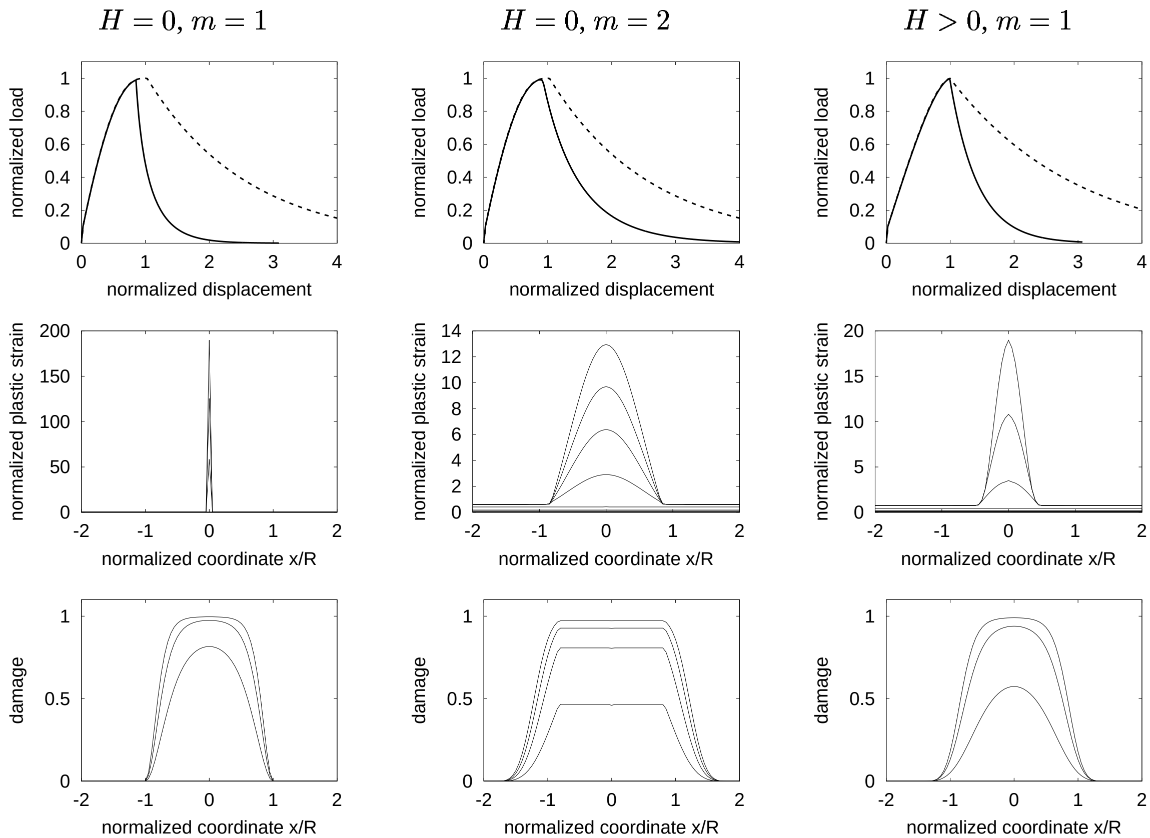

Figure 3: Load-displacement diagrams and distribution of plastic strain and damage for nonlocal plastic-damage model with different combinations of parameters.

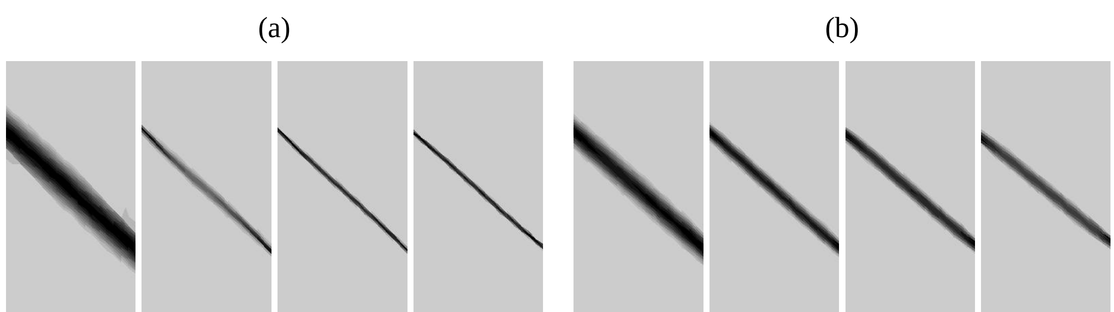

Figure 4: Increments of plastic strain at four stages of analysis; nonlocal plastic-damage model with parameters (a) $m = 1$ and $R = 0.1$ m, (b) $m = 2$ and $R = 0.05$ m.

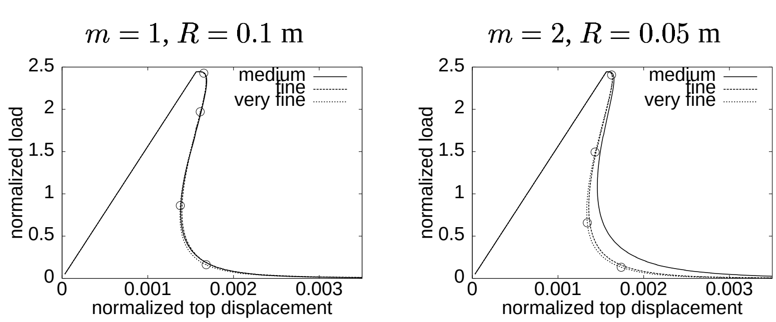

Figure 5: Load-displacement diagrams obtained with nonlocal plastic-damage model on three different meshes.

area under them (dissipated energy) remains nonzero. For comparison, the dashed curves show the load-displacement diagrams for the unstable uniform solution.

Results of two-dimensional analysis of a rectangular panel under plane-strain conditions are presented next. The underlying plasticity model uses the Drucker-Prager yield condition and a nonassociated flow rule, and the damage evolution law is of the exponential type. The hardening law corresponds to perfect plasticity, i.e., $\sigma_Y = \sigma_0 =$ const., which means that the actual plastic modulus H vanishes. The evolution of the active part of plastic zone is shown in Fig. 4a for $m = 1$ and $R = 0.1$ m and in Fig. 4b for $m = 2$ and $R = 0.05$ m. The plotted quantity is the increment of hardening variable κ (scalar measure of plastic strain increment). For $m = 1$, the plastic zone has initially a certain width and later is reduced to a single element layer. For $m = 2$, the active part of the plastic zone also decreases but remains larger than the size of one element (for fine meshes). The load-displacement diagrams in Fig. 5 show very little mesh sensitivity.

5 CONCLUSIONS

We have shown that there is a close relationship between the critical plastic modulus obtained by classical localization analysis based on the acoustic tensor of a local model and the conditions of bifurcation from a uniform state for the nonlocal extension of that model based on nonlocal hardening law or nonlocal damage evolution law. The results serve as a basis for the development of a regularized model for concrete that can capture both tensile and compressive failure modes.

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